Gravitation

Newton's Law of Gravitation

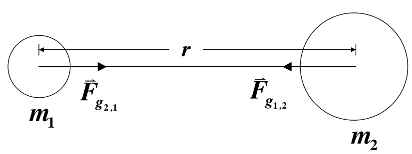
Newton proposed that any two masses were attracted by a gravitational force (inverse square law).

$$F_g = G\frac{m_1 m_2}{r^2} \qquad \boxed{\left|\vec{F}_G\right| = \frac{Gm_1 m_2}{r^2}}$$

Where:

- F = gravitational force (N)
- m = mass of body (kg)
- r = distance between m_1 and m_2 (m)
- $G = 6.67 \times 10^{-11} \left(\frac{\mathrm{N \cdot m^2}}{\mathrm{kg^2}}\right)$

Newton's Law of Gravitation

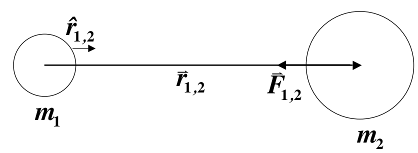


$$F_{g_{1,2}} = F_{g_{2,1}} = G\frac{m_1 m_2}{r^2}$$

Newton's Law of Gravitation (Vector Form)

$$\vec{F}_{1,2} = -G\frac{m_1 m_2}{r^2}\hat{r}_{1,2}$$

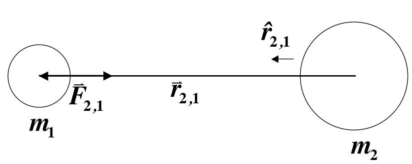
where $\hat{r}_{1,2}$ is a unit vector directed from m_1 to m_2



Newton's Law of Gravitation (Vector Form)

$$\vec{F}_{2,1} = -G\frac{m_1 m_2}{r^2}\hat{r}_{2,1}$$

where $\hat{r}_{2,1}$ is a unit vector directed from m_2 to m_1



Weight

The acceleration of an object (*m*) *due* to a planet or moon's (*M*) gravitation can be found by using the inverse square law and Newton's second law.

$$F_g = G\frac{Mm}{R^2} = mg$$

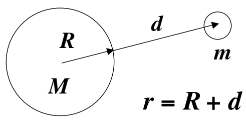
So the acceleration due to gravity at the surface of the planet or moon is:

$$g = \frac{GM}{R^2}$$

Weight

At a point above a planet or moon's surface a distance *r* from the center of the planet or moon the weight of a body is:

$$F_g = G\frac{Mm}{r^2}$$



$$r = R + d$$

Motion of Satellites (Circular Orbits)

A satellite in an orbit that is always the same height above the planet or moon moves with uniform circular motion. Using Newton's second law (*F = ma*):

$$F_g = ma_c$$

$$G\frac{Mm}{r^2} = \frac{mv^2}{r}$$

The *orbital speed* is therefore:

$$v = \sqrt{\frac{GM}{r}}$$

Motion of Satellites (Circular Orbits)

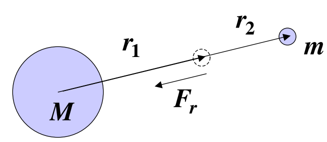
For circular orbits period *T* for the satellite is related to speed *v*.

$$v = \frac{2\pi r}{T}$$

Therefore the period is:

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{GM}} = 2\pi\sqrt{\frac{r^3}{GM}}$$

Gravitational Potential Energy



$$W_{grav} = \int_{r_1}^{r_2} F_r\,dr$$

Because the gravitational force is an attractive force (directed toward the center of the planet or moon) the radial component is negative.

$$F_r = -G\frac{Mm}{r^2}$$

Gravitational Potential Energy

Therefore W_{grav} is given by:

$$W_{grav} = \int_{r_1}^{r_2}\left(-G\frac{Mm}{r^2}\right)dr = -GMm\int_{r_1}^{r_2}\frac{dr}{r^2}$$

$$= -GMm\left(\frac{-1}{r}\right)\Bigg]_{r_1}^{r_2} = -GMm\left(\frac{-1}{r_2} - \frac{-1}{r_1}\right)$$

$$W_{grav} = GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

Gravitational Potential Energy

$$W_{grav} = \frac{GMm}{r_2} - \frac{GMm}{r_1} = -\left(\frac{-GMm}{r_2} - \frac{-GMm}{r_1}\right)$$

Recalling that $W = -\Delta U = -(U_2 - U_1)$

the *gravitational potential energy* is:

$$U = -G\frac{Mm}{r} \qquad U_G = -G\frac{m_1 m_2}{r}$$

***U* is zero when the mass *m* is infinitely far away from the planet or moon.**

Motion of Satellites

The total mechanical energy E of a satellite in a circular orbit of radius r is:

$$E = K + U = \frac{1}{2}mv^2 + \left(-G\frac{Mm}{r}\right)$$

$$E = \frac{1}{2}m\left(\frac{GM}{r}\right) + \left(-G\frac{Mm}{r}\right)$$

$$\boxed{E = -G\frac{Mm}{2r}}$$

Kepler's Laws of Planetary Motion

1.) The paths of the planets are ellipses with the center of the Sun at one focus.

2.) An imaginary line from the Sun to a planet sweeps out equal areas in equal time intervals. Thus, planets move fastest when closest the Sun, slowest when farthest away.

3.) The ratio of the squares of the periods of any two planets revolving about the Sun is equal to the cubes of their average distances from the Sun.

$$\left(\frac{T_a}{T_b}\right)^2 = \left(\frac{r_a}{r_b}\right)^3$$

Kepler's Third Law

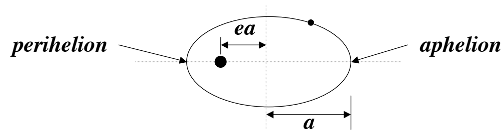
$$\left(\frac{T_a}{T_b}\right)^2 = \left(\frac{r_a}{r_b}\right)^3$$

Recall that for circular orbits:

$$T = 2\pi\sqrt{\frac{r^3}{GM}}$$

$$\boxed{T^2 = \frac{4\pi^2}{GM}r^3}$$

Satellites in an Elliptical Orbit



The point in the planet's orbit closest to the Sun is the *perihelion*, and the point most distant from the Sun is the *aphelion*.

The longest dimension is the *major axis*, with half-length a. This half-length is called the *semi-major axis*.

The distance of each focus from the center of the ellipse is ea where e is the eccentricity.

Satellites in an Elliptical Orbit

The expressions for period T and total energy E for satellites in circular orbits of radius r also hold for elliptical orbits, if r is replaced by a, the length of the semi-major axis:

$$T = 2\pi\sqrt{\frac{a^3}{GM}} \qquad E = -G\frac{Mm}{2a}$$

Escape Velocity

Energy considerations can be used to determine the minimum initial speed needed to allow an object to escape a planet or moon's gravitational field.

At the surface of the planet or moon, $v = v_i$ and $r = r_i = R$. When the object reaches its maximum altitude, $v = v_f = 0$ and $r = r_f = r_{max}$.

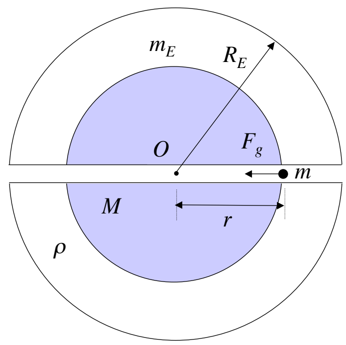
Since total energy is constant:

$$E = K + U = \frac{1}{2}mv_i^2 + \left(-G\frac{Mm}{R}\right) = -G\frac{Mm}{r_{max}}$$

$$v_i^2 = 2GM\left(\frac{1}{R} - \frac{1}{r_{max}}\right)$$

Letting $r_{max} \to \infty$ and taking $v_i = v_{esc}$ $\boxed{v_{esc} = \sqrt{\frac{2GM}{R}}}$

Gravity Inside of the Earth



Gravity Inside of the Earth

Assuming a uniform mass density $\rho = \dfrac{m_E}{V_E} = \dfrac{m_E}{\frac{4}{3}\pi R_E^3}$

$$M = \rho V_M = \left(\frac{m_E}{\frac{4}{3}\pi R_E^3}\right)\frac{4}{3}\pi \cdot r^3 = m_E\,\frac{r^3}{R_E^3}$$

$$F_r = -G\frac{Mm}{r^2} = -G\frac{\left(m_E\,\frac{r^3}{R_E^3}\right)m}{r^2}$$

$$F_r = -G\frac{m_E m}{R_E^3}\,r$$

Gravity Inside of the Earth