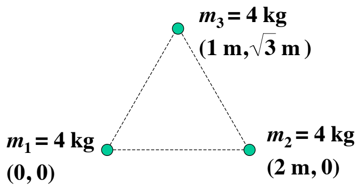
Example 1:

Three point masses are held together by light rods as shown below.



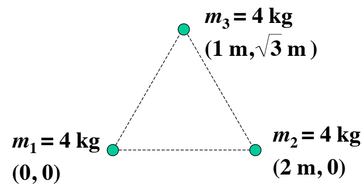
Find:

a.) I_{cm}

b.) I about an axis passing through m_1 and perpendicular to the plane of this arrangement.

Example 1:

a.) I_{cm}

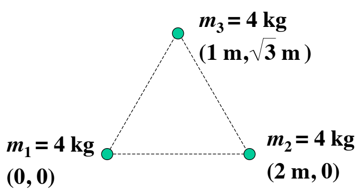


$$x_{cm} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

$$x_{cm} = \frac{(4\text{ kg})(0\text{ m}) + (4\text{ kg})(2\text{ m}) + (4\text{ kg})(1\text{ m})}{4\text{ kg} + 4\text{ kg} + 4\text{ kg}} = 1\text{ m}$$

Example 1:

a.) I_{cm}

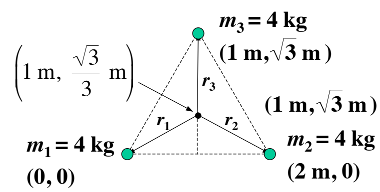


$$y_{cm} = \frac{\sum_i m_i y_i}{\sum_i m_i} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}$$

$$y_{cm} = \frac{(4\text{ kg})(0\text{ m}) + (4\text{ kg})(0\text{ m}) + (4\text{ kg})(\sqrt{3}\text{ m})}{4\text{ kg} + 4\text{ kg} + 4\text{ kg}} = \frac{\sqrt{3}}{3}\text{ m}$$

Example 1:

a.) I_{cm}



$$I_{cm} = \sum_i m_i r_i^2$$

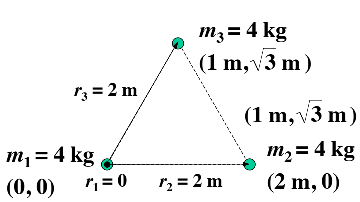
$$r_1 = r_2 = r_3 = \sqrt{(1\text{ m})^2 + \left(\frac{\sqrt{3}}{3}\text{ m}\right)^2} = \sqrt{1\text{ m}^2 + \frac{3}{9}\text{ m}^2} = \sqrt{\frac{12}{9}\text{ m}^2} = \frac{2}{3}\sqrt{3}\text{ m}$$

$$I_{cm} = \sum_i m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 = 3 m_1 r_1^2 = 3(4\text{ kg})\left(\frac{2}{3}\sqrt{3}\text{ m}\right)^2$$

$$\boxed{I_{cm} = 16\text{ kg}\cdot\text{m}^2}$$

Example 1:

b.) I about an axis passing through m_1 and perpendicular to the plane of this arrangement.

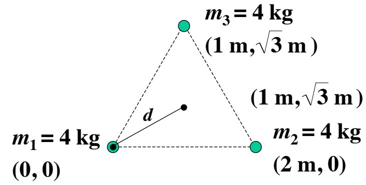


$$I = \sum_i m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 = 0 + 2(2\text{ m})^2(4\text{ kg})$$

$$\boxed{I = 32\text{ kg}\cdot\text{m}^2}$$

Example 1:

b.) I about an axis passing through m_1 and perpendicular to the plane of this arrangement.



or using the Parallel Axis Theorem $d = \frac{2}{3}\sqrt{3}$ m

$$I_p = I_{cm} + Md^2$$

$$I_p = 16\text{ kg}\cdot\text{m}^2 + (12\text{ kg})\left(\frac{2}{3}\sqrt{3}\text{ m}\right)^2$$

$$\boxed{I_p = 32\text{ kg}\cdot\text{m}^2}$$

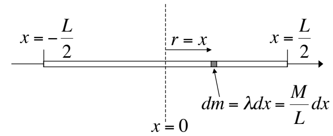
Example 2:

For a uniform rod with a mass M and length L find:

a.) I_{cm}

b.) I about one of its ends.

Example 2a:

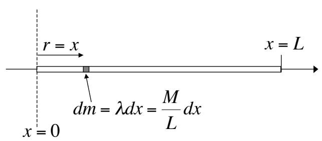


$$I = \int r^2 dm = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \frac{M}{L} dx = \frac{M}{L} \frac{x^3}{3}\Big|_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{M}{L}\left(\frac{\left(\frac{L}{2}\right)^3}{3} - \frac{\left(-\frac{L}{2}\right)^3}{3}\right)$$

$$I = \frac{M}{3L}\left(\frac{L^3}{8} - \left(-\frac{L^3}{8}\right)\right) = \frac{M}{3L}\left(\frac{2L^3}{8}\right)$$

$$\boxed{I = \frac{1}{12} ML^2}$$

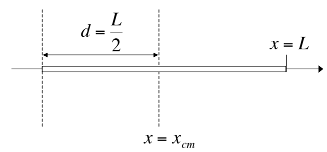
Example 2b:



$$I = \int r^2 dm = \int_0^L x^2 \frac{M}{L} dx = \frac{M}{L} \frac{x^3}{3}\Big|_0^L = \frac{M}{L}\left(\frac{L^3}{3} - \frac{0^3}{3}\right)$$

$$\boxed{I = \frac{1}{3} ML^2}$$

Example 2b:



or using the Parallel Axis Theorem

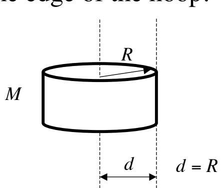
$$I_p = I_{cm} + Md^2$$

$$I_p = \frac{1}{12} ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{12} ML^2 + \frac{1}{4} ML^2$$

$$\boxed{I = \frac{1}{3} ML^2}$$

Example 3:

For a uniform hoop with a mass M and radius R find I about an axis perpendicular to the plane of the hoop and passing through the edge of the hoop.



$$I_{cm} = MR^2$$

$$I_p = I_{cm} + Md^2$$

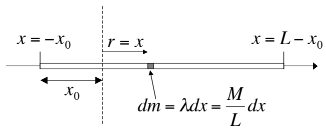
$$I_p = MR^2 + MR^2$$

$$\boxed{I_p = 2MR^2}$$

Example 4:

Show that the minimum moment of inertia for a uniform rod of mass M and length L is located at its center of mass.

Example 4:



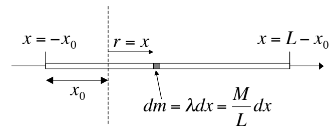
$$I = \int r^2 dm = \int_{-x_0}^{L-x_0} x^2 \frac{M}{L} dx = \frac{M}{L}\frac{x^3}{3}\Big|_{-x_0}^{L-x_0} = \frac{M}{L}\left(\frac{(L-x_0)^3}{3} - \frac{(-x_0)^3}{3}\right)$$

$$I = \frac{M}{3L}\left((L-x_0)^3 + {x_0}^3\right)$$

Minimum occurs when $\frac{dI}{dx_0} = 0$

$$\frac{dI}{dx_0} = \frac{d}{dx_0}\left(\frac{M}{3L}\left((L-x_0)^3 + {x_0}^3\right)\right) = \frac{M}{3L}\left(3(L-x_0)^2(-1) + 3{x_0}^2\right)$$

Example 4:



$$\frac{dI}{dx_0} = \frac{M}{3L}\left(3(L-x_0)^2(-1) + 3{x_0}^2\right) = 0$$

$$\frac{M}{L}\left(\left(L^2 - 2Lx_0 + {x_0}^2\right)(-1) + {x_0}^2\right) = 0$$

$$\frac{M}{L}\left(-L^2 + 2Lx_0 - {x_0}^2 + {x_0}^2\right) = 0$$

$$-L^2 + 2Lx_0 = 0$$

$$2Lx_0 = L^2$$

$$\boxed{x_0 = \frac{L}{2}}$$

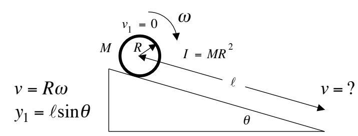
Example 5:

A thin hoop with a radius R and mass M rolls down an incline plane that makes an angle θ with respect to the horizontal. The hoop is released from rest and rolls without slipping a distance l down the incline.

a.) Use energy conservation to find the speed of the hoop.

b.) Use Newton's 2nd Law to find the acceleration of the hoop as it rolls down the incline..

Example 5a:



$$\cancel{K_1} + U_{g_1} = K_2 + \cancel{U_{g_2}}$$

$v_1 = 0$ $\quad$ $y_2 = 0$

$$Mgy_1 = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

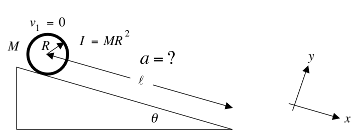
$$Mg\ell\sin\theta = \frac{1}{2}Mv^2 + \frac{1}{2}MR^2\left(\frac{v}{R}\right)^2$$

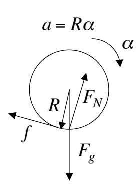
$$g\ell\sin\theta = \frac{1}{2}v^2 + \frac{1}{2}v^2$$

$$g\ell\sin\theta = v^2$$

$$\boxed{v = \sqrt{g\ell\sin\theta}}$$

Example 5b:





$\sum F_x = ma$

$F_{//} - f = Ma$

$F_g\sin\theta - f = Ma$

(1) $\quad Mg\sin\theta - f = Ma$

$\sum F_y = ma$

$F_N - F_\perp = 0$

$F_N = Mg\cos\theta$

$\sum \tau = I\alpha$

$\tau_f = I\alpha$

$Rf\sin\theta_f = MR^2\alpha$

$Rf(1) = MR^2\left(\frac{a}{R}\right)$

(2) $\quad f = Ma$

$(1+2) \quad Mg\sin\theta = 2Ma$

$$\boxed{a = \frac{g\sin\theta}{2}}$$

Example 5:

Check this out.

Since the acceleration down the incline is uniform:

$$v^2 = {v_1}^2 + 2a\Delta x$$

$$v^2 = 0 + 2\left(\frac{g\sin\theta}{2}\right)\ell$$

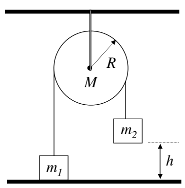
$$v^2 = g(\sin\theta)\ell$$

$$\boxed{v = \sqrt{g\ell\sin\theta}}$$

This is consistent to what was found using energy.

Example 6

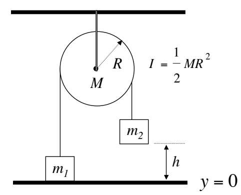
In the figure below, the pulley is a solid disk of mass M and radius R. Two blocks, one of mass m_1 and one of mass $m_2 > m_1$, hang from either side of the pulley by a light cord. Initially the system is at rest, with m_1 on the floor and m_2 held at height h above the floor. Then m_2 is released and allowed to fall.

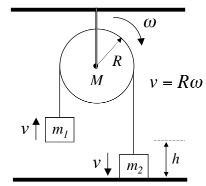


a.) Use energy conservation to find the speed of Block 2 just before it strikes the ground.

b.) Use Newton's 2nd Law to find the acceleration of Block 2 as it falls to the floor.

Example 6a:





$$\cancel{K_1} + U_{g_1} = K_2 + U_{g_2}$$

$$v_1 = 0$$

$$m_2 g y_1 = \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}I\omega^2 + m_1 g y_2$$

$$m_2 gh = \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2 + m_1 gh$$

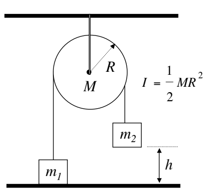
$$m_2 gh = \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{4}Mv^2 + m_1 gh$$

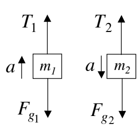
$$m_2 gh - m_1 gh = \left(\frac{1}{2}(m_1 + m_2) + \frac{1}{4}M\right)v^2$$

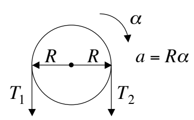
$$v^2 = \frac{(m_2 - m_1)gh}{\frac{1}{2}(m_1 + m_2) + \frac{1}{4}M}$$

$$\boxed{v = \sqrt{\frac{(m_2 - m_1)gh}{\frac{1}{2}(m_1 + m_2) + \frac{1}{4}M}}}$$

Example 6b:







$$\sum F_y = ma$$

$$(1)\ T_1 - F_{g_1} = m_1 a$$

$$(2)\ F_{g_2} - T_2 = m_2 a$$

$$\sum \tau = I\alpha$$

$$\tau_{T_1} + \tau_{T_2} = I\alpha$$

$$RT_1 \sin\theta_{T_1} + RT_2 \sin\theta_{T_2} = \frac{1}{2}MR^2\alpha$$

$$RT_1(-1) + RT_2(1) = \frac{1}{2}MR^2\left(\frac{a}{R}\right)$$

$$(3)\ T_2 - T_1 = \frac{1}{2}Ma$$

$$(1+2+3)\ F_{g_2} - F_{g_1} = m_1 a + m_2 a + \frac{1}{2}Ma$$

$$m_2 g - m_1 g = \left(m_1 + m_2 + \frac{1}{2}M\right)a$$

$$\boxed{a = \frac{(m_2 - m_1)g}{m_1 + m_2 + \frac{1}{2}M}}$$

Example 6:

Check this out.

Since the acceleration is uniform:

$$v^2 = v_1^2 + 2a\Delta y$$

$$v^2 = 0 + 2\left(\frac{(m_2 - m_1)g}{m_1 + m_2 + \frac{1}{2}M}\right)h$$

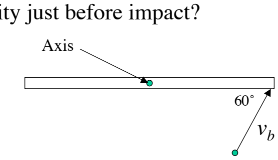
$$v^2 = \frac{2(m_2 - m_1)gh}{m_1 + m_2 + \frac{1}{2}M} = \frac{(m_2 - m_1)gh}{\frac{1}{2}m_1 + \frac{1}{2}m_2 + \frac{1}{4}M}$$

$$\boxed{v = \sqrt{\frac{(m_2 - m_1)gh}{\frac{1}{2}(m_1 + m_2) + \frac{1}{4}M}}}$$

This is consistent to what was found using energy.

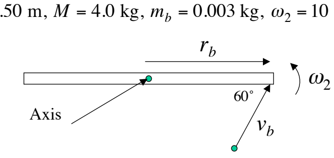
Example 7

A uniform thin rod of length 0.50 m and mass 4.0 kg can rotate in a horizontal plane about a vertical axis through its center. The rod is at rest when a 3.0 g bullet traveling in the horizontal plane of the rod is fired into one end of the rod. As viewed from above, the direction of the bullet's velocity makes an angle of 60° with the rod. If the bullet lodges in the rod and the angular velocity of the rod is 10 rad/s immediately after the collision, what is the magnitude of the bullet's velocity just before impact?



Example 7:

$$\ell = 0.50\ \text{m},\ M = 4.0\ \text{kg},\ m_b = 0.003\ \text{kg},\ \omega_2 = 10\ \frac{\text{rad}}{\text{s}},\ v_b = ?$$



$$\sum L_i = \sum L_f$$

$$L_{bullet_1} = L_{bullet_2} + L_{Rod_2}$$

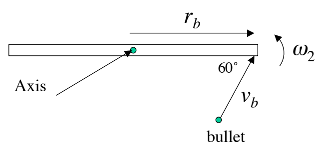
$$\vec{r} \times \vec{p}_b = I_2\omega_2$$

$$r_b m_b v_b \sin\theta_b = (I_{Rod} + I_{bullet})\omega_2$$

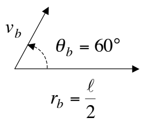
$$r_b m_b v_b \sin\theta_b = \left(\frac{1}{12}M\ell^2 + m_b r_b^2\right)\omega_2$$

Example 7:

$$\ell = 0.50\text{ m},\ M = 4.0\text{ kg},\ m_b = 0.003\text{ kg},\ \omega_2 = 10\ \frac{\text{rad}}{\text{s}},\ v_b = ?$$

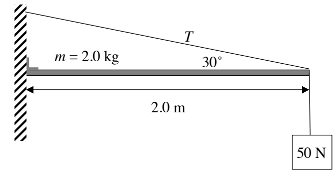


$$r_b m_b v_b \sin\theta_b = \left(\frac{1}{12}M\ell^2 + m_b r_b^{\,2}\right)\omega_2$$



$$v_b = \frac{\left(\frac{1}{12}M\ell^2 + m_b r_b^{\,2}\right)\omega_2}{r_b m_b \sin\theta_b} = \frac{\left(\frac{1}{12}M\ell^2 + m_b\left(\frac{\ell}{2}\right)^2\right)\omega_2}{\frac{\ell}{2} m_b \sin\theta_b}$$

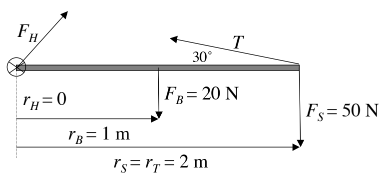
$$v_b = \frac{\left(\frac{1}{12}(4.0\text{ kg})(0.5\text{ m})^2 + (0.003\text{ kg})\left(\frac{0.5\text{ m}}{2}\right)^2\right)\left(10\ \frac{\text{rad}}{\text{s}}\right)}{\left(\frac{0.5\text{ m}}{2}\right)(0.003\text{ kg})\sin 60°} = \boxed{1286\ \frac{\text{m}}{\text{s}}}$$



Example 8:

A 50 N sign is attached to the end of a 2 m horizontal beam that is hinged to a vertical support. The beam is uniform with a mass of 2.0 kg. A cable that makes an angle of 30° with respect to the beam is attached to the wall to help support the sign. Find the tension in the cable and the force exerted on the beam by the vertical support.

Example 8:



$$\sum \tau = 0$$

$$\tau_H + \tau_B + \tau_S + \tau_T = 0 \quad (r_H = 0)$$

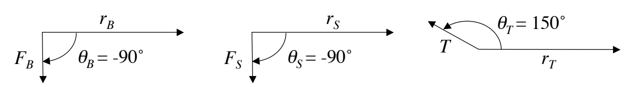
$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \tau = rF\sin\theta$$

$$r_B F_B \sin\theta_B + r_S F_S \sin\theta_S + r_T T \sin\theta_T = 0$$

$$T = \frac{-r_B F_B \sin\theta_B - r_S F_S \sin\theta_S}{r_T \sin\theta_T}$$

Example 8:

$$T = \frac{-r_B F_B \sin\theta_B - r_S F_S \sin\theta_S}{r_T \sin\theta_T}$$



$$T = \frac{-(1\text{ m})(20\text{ N})\sin(-90°) - (2\text{ m})(50\text{ N})\sin(-90°)}{(2\text{ m})\sin(150°)} = \boxed{120\text{ N}}$$

$$\sum F_x = 0$$

$$F_{H_x} + T_x = 0$$

$$F_{H_x} = -T_x = -T\cos\theta_T = -(120\text{ N})\cos(150°)$$

$$F_{H_x} = 103.9\text{ N}$$

Example 8:

$$\sum F_y = 0$$

$$F_{H_y} - F_B - F_S + T_y = 0$$

$$F_{H_y} = F_B + F_S - T_y = 20\text{ N} + 50\text{ N} - (120\text{ N})\sin(150°)$$

$$F_{H_y} = 10.0\text{ N}$$

$$\vec{F}_H = (103.9\text{ N})\hat{i} + (10.0\text{ N})\hat{j}$$

$$F_H = \sqrt{F_x^{\,2} + F_y^{\,2}} = \sqrt{(103.9\text{ N})^2 + (10.0\text{ N})^2} = 104.4\text{ N}$$

$$\theta_H = \tan^{-1}\frac{F_y}{F_x} = \tan^{-1}\left(\frac{10.0}{103.9}\right) = 5.5°$$

$$\boxed{\vec{F}_H = 104.4\text{ N}\angle 5.5°}$$