Oscillations

Oscillations

An *oscillation* is a repetitive motion about an equilibrium position. This is also referred to as *periodic motion*.

The *amplitude A* of the motion is the maximum displacement from the equilibrium position.

The *period T* is the time for one cycle of motion.

The *frequency f* is the number of cycles in a unit of time.

The *angular frequency* ω is 2π times the frequency.

$$f = \frac{1}{T} \qquad \omega = 2\pi f = \frac{2\pi}{T} \qquad \boxed{T = \frac{2\pi}{\omega} = \frac{1}{f}}$$

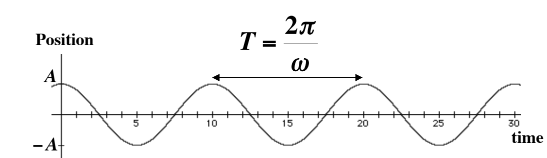
Simple Harmonic Motion

Simple Harmonic Motion (***SHM***) is periodic motion in which the restoring force is directly proportional to the displacement from the equilibrium position.

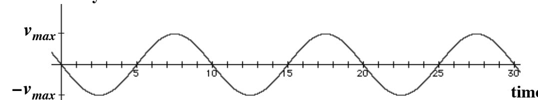
$$F_x = -kx$$

Simple Harmonic Motion is an oscillation that is described by a sinusoidal function.

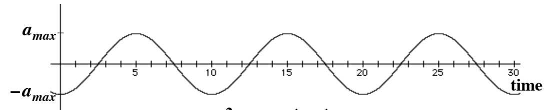
$$\boxed{x = A\cos(2\pi f t)} = A\cos(\omega t)$$



$$x = A\cos(\omega t)$$



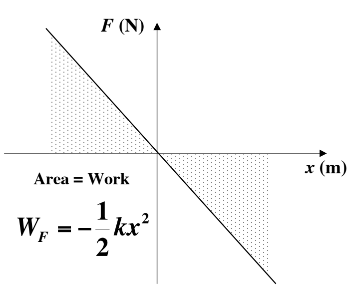
$$v = -\omega A\sin(\omega t) = -v_{max}\sin(\omega t)$$



$$a = -\omega^2 A\cos(\omega t) = -a_{max}\cos(\omega t)$$

Potential Energy in SHM

$$F_x = -kx$$



$$W_F = -\frac{1}{2}kx^2$$

$$U_F = -W_F = \frac{1}{2}kx^2$$

Energy in Simple Harmonic Motion

The total energy is the sum of the kinetic and potential energies of the object.

$$E = K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$

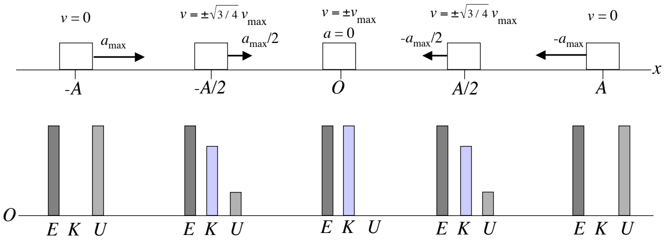
$$E = \tfrac{1}{2}m\left(-\omega A\sin(\omega t)\right)^2 + \tfrac{1}{2}k\left(A\cos(\omega t)\right)^2$$

$$E = \tfrac{1}{2}kA^2\sin^2(\omega t) + \tfrac{1}{2}kA^2\cos^2(\omega t)$$

$$E = \tfrac{1}{2}kA^2\left(\sin^2(\omega t) + \cos^2(\omega t)\right)$$

$$\boxed{E = \tfrac{1}{2}kA^2}$$

Energy in Periodic Motion



Maximum Speed

At the maximum displacements all of energy is potential energy and:

$$K = 0 \text{ and } U = \tfrac{1}{2}kA^2$$

Passing through the equilibrium point all of the energy is kinetic and:

$$K = \tfrac{1}{2}kA^2 \text{ and } U = 0$$

Since the maximum speed occurs at equilibrium:

$$K = \tfrac{1}{2}mv_{max}^2 = \tfrac{1}{2}kA^2 \text{ and } \boxed{v_{max} = \sqrt{\frac{k}{m}}A}$$

Period of Oscillation

Since $v = -\omega A \sin(\omega t) = -v_{max}\sin(\omega t)$ **it follows that :**

$$v_{max} = \omega A = \sqrt{\frac{k}{m}}A \text{ and } \boxed{\omega = \sqrt{\frac{k}{m}}}$$

The period T is therefore:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{k}{m}}}$$

$$\boxed{T_s = 2\pi\sqrt{\frac{m}{k}}}$$

Maximum Acceleration

At the maximum acceleration occurs when the restoring force is a maximum. This occurs when the displacement is A.

$$F = kx = kA = ma_{max}$$

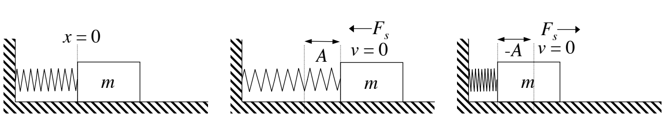
$$\boxed{a_{max} = \frac{k}{m}A = \omega^2 A}$$

Horizontal Motion of a Mass on a Spring

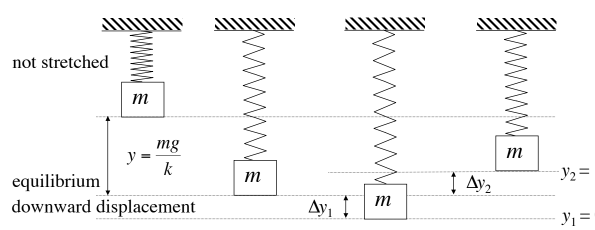
The restoring force for an ideal spring is:

$$F_s = -kx$$

An attached mass m will undergo SHM when displaced from equilibrium on a frictionless surface.



Vertical Spring Oscillation Situation 1:



$$K_1 + U_{e_1} + U_{g_1} = K_2 + U_{e_2} + U_{g_2}$$

$$U_{e_1} = U_{e_2} + U_{g_2}$$

$$\frac{1}{2}k(y + \Delta y_1)^2 = \frac{1}{2}k(y - \Delta y_2)^2 + mg(\Delta y_1 + \Delta y_2)$$

Vertical Spring Oscillation Situation 1:

$$y^2 + 2y\Delta y_1 + \Delta y_1^2 = y^2 - 2y\Delta y_2 + \Delta y_2^2 + \frac{2mg}{k}(\Delta y_1 + \Delta y_2)$$

$$2y\Delta y_1 + \Delta y_1^2 = -2y\Delta y_2 + \Delta y_2^2 + \frac{2mg}{k}\Delta y_1 + \frac{2mg}{k}\Delta y_2$$

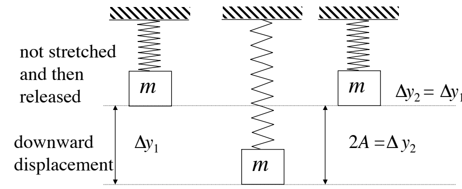
$$2\frac{mg}{k}\Delta y_1 + \Delta y_1^2 = -2\frac{mg}{k}\Delta y_2 + \Delta y_2^2 + \frac{2mg}{k}\Delta y_1 + \frac{2mg}{k}\Delta y_2$$

$$\Delta y_1^2 = \Delta y_2^2$$

$$\Delta y_1 = \Delta y_2 \quad \text{and} \quad A = \Delta y_1$$

The motion has an amplitude $A = \Delta y_1$ with an equilibrium point located at $y = \dfrac{mg}{k}$.

Vertical Spring Oscillation Situation 2:



$$K_1 + U_{e_1} + U_{g_1} = K_2 + U_{e_2} + U_{g_2}$$

$$U_{g_1} = U_{e_2}$$

$$mg\Delta y_1 = \frac{1}{2}k\Delta y_1^2$$

Vertical Spring Amplitude Situation 2:

$$mg\Delta y_1 = \frac{1}{2}k\Delta y_1^2$$

$$\frac{2mg}{k} = \Delta y_1$$

$$\Delta y_1 = 2A$$

$$A = \frac{mg}{k}$$

The motion has an amplitude $A = \dfrac{mg}{k}$ with an equilibrium point also located at $y = \dfrac{mg}{k}$.

Motion of a Mass on a Vertical Spring

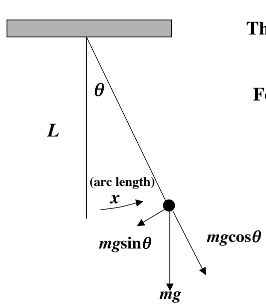
A mass m on a vertical spring oscillates with simple harmonic motion with its equilibrium position determined by the force of gravity on the mass. At equilibrium:

$$F_g = mg = ky_0$$

$$y_0 = \frac{mg}{k}$$

The Simple Pendulum

A *simple pendulum* is an idealized model consisting of a point mass suspended by a massless, unstretchable string.



The restoring force F is:

$$F = -mg\sin\theta$$

For small displacements $\sin\theta \approx \theta$ and

$$F = -mg\theta$$

$$F = -mg\theta = -mg\frac{x}{L}$$

$$F = -\frac{mg}{L}x$$

The Simple Pendulum

$$F = -\frac{mg}{L}x = -kx$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg/L}{m}} = \sqrt{\frac{g}{L}}$$

The period T of a simple pendulum for small amplitudes is:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \qquad \boxed{T_p = 2\pi\sqrt{\frac{\ell}{g}}}$$