Rotation of Rigid Bodies

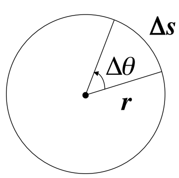
Rotation of Rigid Bodies

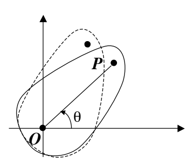
- A *rigid body* has a definite size and shape.
- Forces that act on them do not cause deformations such as stretching, twisting, and squeezing.
- The motions cannot be described as a moving point. Each involves a body that rotates about an axis that is stationary in some inertial frame of reference.

Angle Measurement

For rotational motion, the most natural way to measure angles is radians. The value of $\Delta\theta$ (in radians) is equal to the arc length Δs divided by the radius r. The coordinate θ specifies the rotational position of a rigid body at a given instant.

$$\Delta s = r\Delta\theta$$





Angular Velocity

The *average angular velocity* is:

$$\omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

The *instantaneous angular velocity* is:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

At any instant, every part of a rotating rigid body has the same angular velocity moving through different distances.

Angular Acceleration

The *average angular acceleration* is:

$$\alpha_{av} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

The *instantaneous angular acceleration* is:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$$

At any instant, every part of a rotating rigid body has the same angular acceleration.

Constant Angular Acceleration

The *angular acceleration* is:

$$\alpha = \frac{\omega - \omega_0}{t - 0} \quad \text{or} \quad \boxed{\omega = \alpha t + \omega_0}$$

The *average angular velocity* is:

$$\omega_{av} = \frac{\omega_0 + \omega}{2} = \frac{\Delta\theta}{\Delta t}$$

Using the relationship between angular velocity and total angular displacement the above equation becomes:

$$\Delta\theta = \left(\frac{\omega + \omega_0}{2}\right)t$$

Constant Angular Acceleration

The equations used to describe linear motion all have rotational equivalents.

$$\omega = \alpha t + \omega_0$$

$$\theta = \left(\frac{\omega + \omega_0}{2}\right)t + \theta_0 \qquad \Delta\theta = \left(\frac{\omega + \omega_0}{2}\right)t$$

$$\theta = \tfrac{1}{2}\alpha t^2 + \omega_0 t + \theta_0 \qquad \Delta\theta = \tfrac{1}{2}\alpha t^2 + \omega_0 t$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \qquad \omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

Relating Linear and Angular Kinematics

- **Points on a rotating body all rotate at the same angular velocity.**
- **The tangential (linear) speed of a point depends upon its distance from the axis of rotation.**
- **The tangential speed of a particle is directly proportional to its angular velocity.**

$$\Delta s = r\Delta\theta$$

$$\frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$$

$$v_t = r\omega$$

(θ must be in radians)

Relating Linear and Angular Kinematics

- **All points on a rotating object have the same angular acceleration.**
- **The linear (tangential) acceleration is related to the angular acceleration and is tangent to the circular path.**

$$a_t = \frac{\Delta v}{\Delta t} = \frac{\Delta r\omega}{\Delta t} = r\frac{\Delta\omega}{\Delta t}$$

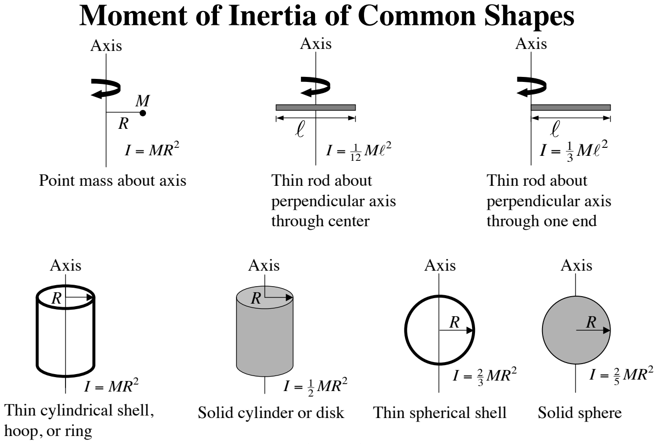
$$a_t = r\alpha$$

(θ must be in radians)

Moment of Inertia (I)

- **The resistance of an object to changes in rotational motion is called the *moment of inertia*.**
- ***Moment of inertia* is the rotational analog of mass.**
- **The *moment of inertia* depends upon an object's mass and its distribution around the axis of rotation.**
- **The units are (kg·m^2).**

Moment of Inertia of Common Shapes



Rotational Kinetic Energy

In terms of moment of inertia I, the *rotational kinetic energy* of a rigid body is:

$$K = \tfrac{1}{2}I\omega^2$$

Center of Mass

- **The *center of mass* is the point around which an object rotates if gravity is the only force acting on the object.**
- **The *center of mass* is also the point at which all the mass of body can be considered.**
- **For regularly shaped objects (spheres, cubes, bars) the *center of mass* is at the geometric center of the object.**

Total Kinetic Energy of Rolling Objects

The motion of an object can always be divided into *independent* translation of the center of mass (*cm*) ands rotation about the center of mass.

$$K = \tfrac{1}{2}Mv_{cm}^2 + \tfrac{1}{2}I\omega^2$$

(translation) (rotation)

When an object is *rolling without slipping*:

$$v_{cm} = R\omega$$

Mechanical Energy Conservation with Rotation

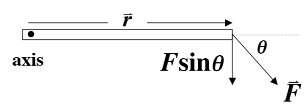
$$E = K_{trans} + K_{rot} + U$$

$$E = \tfrac{1}{2}mv_{cm}^2 + \tfrac{1}{2}I\omega^2 + mgy$$

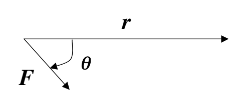
$$E_i = E_f$$

Torque (τ)

- ***Torque* is the ability of a *force to rotate* an object around some axis.**
- **The units are (N·m).**



$$\tau = r \cdot F\sin\theta$$



Torque

The sign of the torque resulting from a force is positive if the rotation is counterclockwise.

The sign of the torque resulting from a force is negative if the rotation is clockwise.

Newton's Second Law for Rotation

The net torque on a rigid body equals the body's moment of inertia about the rotation axis times its angular acceleration.

$$\sum \tau = I\alpha$$

Angular Momentum

- Because a rotating object has inertia, it has momentum associated with its rotation.
- This momentum is called *angular momentum*.

$$\boxed{L = I\omega}$$

- The units are (kg·m²/s)

More on Angular Momentum

$$\sum \tau = I\alpha$$

$$\sum \tau = I\frac{\Delta\omega}{\Delta t}$$

$$I\Delta\omega = \sum \tau \cdot \Delta t$$

$$\Delta L = \sum \tau \cdot \Delta t$$

$$\boxed{\Delta L = \tau\Delta t}$$

Conservation of Angular Momentum

When the net external torque acting on an object or objects is zero, the angular momentum of the object(s) does not change.

$$\Delta L = \tau\Delta t = 0$$

$$\sum L_i = \sum L_f$$

$$\sum I_i\omega_i = \sum I_f\omega_f$$

Linear and Rotational Analogs

Linear Quantity	Rotational Analog
$x,\ y,$ or $s = (\text{m})$	$\theta = (\text{rad})$
$v = (\text{m/s})$	$\omega = (\text{rad/s})$
$a = (\text{m/s}^2)$	$\alpha = (\text{rad/s}^2)$
$F = (\text{N})$	$\tau = (\text{N}\cdot\text{m})$
$K = \frac{1}{2}mv^2$	$K_{rot} = \frac{1}{2}I\omega^2$
$p = mv$	$L = I\omega$
$\sum F = ma$	$\sum \tau = I\alpha$
$\Delta p = F\cdot\Delta t$	$\Delta L = \tau\cdot\Delta t$

Equilibrium

First condition for equilibrium (no acceleration)

$$\sum F_x = 0 \text{ and } \sum F_y = 0$$

Second condition for equilibrium (no rotation)

$$\sum \tau = 0$$

The sum of the torques due to all external forces acting on the body, with respect to any specified point, must be zero.