AP Calculus BC Cheatsheet

Fundamentals

Definition of continuity at a point:	$\lim_{x\to c} f(x) = f(c)$	
Definition of the derivative:	$\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$	$\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$
Riemann sum definition of the integral:	$\lim_{\|\|p\|\|\to 0} \sum_{i=0}^{N} f(c_i)\Delta x_i = \int_a^b f(x)dx$	
Fundamental theorem of calculus:	$f(b) = f(a) + \int_a^b f'(x)dx$	$\int f'(x)dx = f(x) + C$
	$\frac{d}{dx}\int_0^x f(t)dt = f(x)$	$\frac{d}{dx}\int_0^{g(x)} f(t)dt = f(g(x))g'(x)$

If $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$, then $\lim_{x\to c} f(x) = L$.

Interpretations of 1st and 2nd Derivatives

	$f'(x)$	$f''(x)$
> 0	$f(x)$ is increasing	$f'(x)$ is increasing $f(x)$ is concave up
< 0	$f(x)$ is decreasing	$f'(x)$ is decreasing $f(x)$ is concave down
changes $+$ to $-$	$f(x)$ is at a maximum	$f'(x)$ is at a maximum $f(x)$ is at a point of inflection
changes $-$ to $+$	$f(x)$ is at a minimum	$f'(x)$ is at a minimum $f(x)$ is at a point of inflection

If $f'(x) = 0$ and $f''(x) > 0$, then $f(x)$ is at a minimum.
If $f'(x) = 0$ and $f''(x) < 0$, then $f(x)$ is at a maximum.

To find the local maximum or minimum of $f(x)$ on the closed interval $[a, b]$, evaluate $f(x)$ at a, b, and all critical values of $f(x)$.

Theorems

Mean Value Theorem: If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then $\exists c \in [a, b] : f'(c) = \frac{f(b)-f(a)}{b-a}$

L'Hopital's Rule: If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is indeterminate $\left(\frac{0}{0}, \pm\frac{\infty}{\infty}, \pm\infty\cdot 0, \infty-\infty, 0^0, 1^\infty, \infty^0\right)$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Integration and Differentiation Methods

Logarithmic Differentiation:

$y = f(x)^{g(x)}$

$\ln(y) = \ln\left(f(x)^{g(x)}\right)$

$\ln(y) = g(x)\ln(f(x))$

$\frac{y'}{y} = g(x)\frac{1}{f(x)}f'(x) + g'(x)\ln(f(x))$

$y' = y\left(\frac{g(x)f'(x)}{f(x)} + g'(x)\ln(f(x))\right)$

$y' = f(x)^{g(x)}\left(\frac{g(x)f'(x)}{f(x)} + g'(x)\ln(f(x))\right)$

Partial Fraction Decomposition:

$\frac{4x^2-7x-3}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$ (note repeated linear factor)

$4x^2-7x-3 = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$

$4x^2-7x-3 = Ax^2 - 2Ax + A + Bx^2 + Bx - 2B + Cx + 2c$

$4x^2-7x-3 = (A+B)x^2 + (-2A+B+C)x + (A-2B+2C)$

$4 = A+B; -7 = -2A+B+C; -3 = A-2B+2C$ (system of equations)

$\frac{4x^2-7x-3}{(x+2)(x-1)^2} = \frac{3}{x+2} + \frac{1}{x-1} + \frac{-2}{(x-1)^2}$

Integration by Parts: $\int u\,dv = uv - \int v\,du$ or $\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$

When integrating by parts, choose u in the order: logs, inverse trig, algebraic, trig, exponential

Trig substitutions:

when given:	let:
$\sqrt{a^2-x^2}$	$x = a\sin(\theta)$
$\sqrt{a^2+x^2}$	$x = a\tan(\theta)$
$\sqrt{x^2-a^2}$	$x = a\sec(\theta)$

Derivatives

constants can be factored out:	$\frac{d}{dx}a\cdot f(x) = a\frac{d}{dx}f(x)$
Power rule if $a = 0$ (f is constant):	$\frac{d}{dx}x^0 = 0$
Power rule if $a \neq 0$:	$\frac{d}{dx}x^a = ax^{a-1}$
Inverse rule:	$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$
Product rule:	$\frac{d}{dx}f(x)g(x) = f(x)g'(x) + f'(x)g(x)$
Quotient rule:	$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - g'(x)f(x)}{(g(x))^2}$
Chain rule:	$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$
Sum/difference rule:	$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

Trig Rules

$\frac{d}{dx}\sin(x) = \cos(x)$

$\frac{d}{dx}\cos(x) = -\sin(x)$

$\frac{d}{dx}\tan(x) = \sec(x)^2$

$\frac{d}{dx}\cot(x) = -\csc(x)^2$

$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$

$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$

Inverse Trig Rules

$\frac{d}{dx}\sin^{-1}(u) = \frac{u'}{\sqrt{1-u^2}}$

$\frac{d}{dx}\cos^{-1}(u) = \frac{-u'}{\sqrt{1-u^2}}$

$\frac{d}{dx}\tan^{-1}(u) = \frac{u'}{1+u^2}$

$\frac{d}{dx}\cot^{-1}(u) = \frac{-u'}{1+u^2}$

$\frac{d}{dx}\sec^{-1}(u) = \frac{u'}{|u|\sqrt{u^2-1}}$

$\frac{d}{dx}\csc^{-1}(u) = \frac{-u'}{|u|\sqrt{u^2-1}}$

Exponential/Log Rules

$\frac{d}{dx}e^x = e^x$

$\frac{d}{dx}a^x = a^x\ln(a)$

$\frac{d}{dx}\ln(x) = \frac{1}{x}$

$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$

Integrals

constants can be factored out $\int a \cdot f(x)dx = a\int f(x)dx$

Power rule if $a=-1$: $\int x^{-1}dx = \int \frac{1}{x}dx = \ln(x) + C$

Power rule if $a \neq -1$: $\int x^a dx = \frac{x^{a+1}}{a+1} + C$

Sum/difference rule: $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$

Trig Rules

$$\int \sin(u)dx = -\cos(u) + C$$

$$\int \cos(u)dx = \sin(u) + C$$

$$\int \sec(u)^2 dx = \tan(u) + C$$

$$\int \tan(u)dx = -\ln(|\cos(u)|) + C$$

$$\int \csc(u)^2 dx = -\cot(u) + C$$

$$\int \cot(u)dx = \ln(|\sin(u)|) + C$$

$$\int \sec(u)\tan(u)dx = \sec(u) + C$$

$$\int \sec(u)dx = \ln(|\tan(u) + \sec(u)|) + C$$

$$\int \csc(u)\cot(u)dx = -\csc(u) + C$$

$$\int \csc(u)dx = -\ln(|\cot(u) + \csc(u)|) + C$$

Exponential/Log Rules

$$\int \frac{1}{x}dx = \ln(x) + C$$

$$\int e^u du = e^u + C$$

$$\int a^u du = \frac{a^u}{\ln(a)} + C$$

Inverse Trig Rules

$$\int \frac{1}{\sqrt{a^2-u^2}}du = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{1}{a^2+u^2}du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{1}{u\sqrt{u^2-a^2}}du = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C$$

Riemann Sum Approximations

Trapezoid: $\frac{b-a}{2n}\sum_{i=0}^{n-1}(f(x_i) + f(x_{i+1})) = \frac{b-a}{2n}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right)$

Lefthanded rectangle: $\sum_{i=0}^{n-1} f(x_i)\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$

Righthanded rectangle: $\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$

Mean Values and Shapes of Revolution

Mean value of $f(x)$ on the closed interval $[a,b]$: $\frac{\int_a^b f(x)dx}{b-a}$

Mean rate of change of $f(x)$ on the closed interval $[a,b]$: $\frac{f(b)-f(a)}{b-a}$

Arclength of $f(x)$ on the closed interval $[a,b]$: $\int_a^b \sqrt{1+(f'(x))^2}dx$

Surface area formed by rotating $f(x)$ from a to b about x axis: $2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}dx$

Volume formed by rotating area between $f(x)$ and $g(x)$ from a to b about x axis (where $|f(x)| > |g(x)|\ \forall x \in [a,b]$): $\pi\int_a^b \left((f(x))^2 - (g(x))^2\right)dx$

Volume formed by rotating area between $f(x)$ and $g(x)$ from a to b about $y=k$ (where $|k-f(x)| > |k-g(x)|\ \forall x \in [a,b]$): $\pi\int_a^b \left((k-f(x))^2 - (k-g(x))^2\right)dx$

Parametrics, Vectors, Polars

Parametrics and Vectors

$\vec{r} = \langle x, y \rangle$

$\vec{v} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$\vec{a} = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle \qquad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

arclength from $t = a$ to $t = b$: $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

direction of $\vec{a}$: $\tan^{-1}\left(\frac{d^2x}{dt^2} \bigg/ \frac{d^2y}{dt^2}\right)$

$\vec{v}$ is tangent to the curve.

$\frac{dy}{dt} = 0 \Rightarrow$ horizontal tangent line.

$\frac{dx}{dt} = 0 \Rightarrow$ vertical tangent line.

Polars

$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad x = r \cdot \cos(\theta) \qquad y = r \cdot \sin(\theta)$

slope at $(r, \theta) = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin(\theta) + r \cdot \cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r \cdot \sin(\theta)}$

$A = \frac{1}{2}\int_a^b (r(\theta))^2 d\theta$ or $A = \frac{1}{2}\int_a^b \left((r_{outer}(\theta))^2 - r_{inner}(\theta))^2\right) d\theta$

Series

Maclaurin (centered at 0): $\sum_{n=0}^{\infty} \frac{f^n(0)x^n}{n!}$

Taylor (centered at c): $\sum_{n=0}^{\infty} \frac{f^n(c)(x-c)^n}{n!}$

Lagrange form of remainder: $R_n(x) = \frac{f^{n+1}(z)(x-c)^{n+1}}{(n+1)!}, z \in (c, x)$

Alternating series remainder where $a_{n+1} \leq a_n$: $|S - S_n| = |R_n| \leq a_{n+1}$

Common Power Series

$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}***$

$\sin^{-1}(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}(2n)!}{(2^{2n})(2n+1)(n!)^2}* \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n*$

$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}* \qquad \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n*$

$(1+x)^k = x^k \sum_{n=0}^{\infty} \frac{x^{-n}k!}{n!(k-n)!}** \qquad (1+x)^k = \sum_{n=0}^{\infty} \frac{x^n k!}{n!(k-n)!}*$

$*|x| < 1 \qquad **|x| > 1 \qquad ***|x-1| < 1$

Convergence Rules

Name	Applicable Form	Converges if	Diverges if	Notes
nth Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	
Integral	$\sum_{n=1}^{\infty} a_n, a_n = f(n) \geq 0, f'(x) < 0$	$\int_1^{\infty} f(n)dn$ converges	$\int_1^{\infty} f(n)dn$ diverges	remainder: $0 < R_N < \int_N^{\infty} f(n)dn$
Geometric	$\sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	sum: $S = \frac{a}{1-r}$
p Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	
Telescoping	$\sum_{n=1}^{\infty} (a_n - a_{n+1})$	$\lim_{n\to\infty} a_n = L < \infty$		sum: $S = a_1 - L$
Alternating	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n, \lim_{n\to\infty} a_n = 0$		remainder: $\lvert R_N\rvert \leq a_{N+1}$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$	
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$	
Direct Comparison	$\sum_{n=1}^{\infty} a_n$	$0 \leq a_n \leq b_n, \sum_{n=1}^{\infty} b_n$ converges	$0 \leq b_n \leq a_n, \sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison	$\sum_{n=1}^{\infty} a_n$	$0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty, \sum_{n=1}^{\infty} b_n$ converges	$0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty, \sum_{n=1}^{\infty} b_n$ diverges	

Miscellaneous

Differentiability implies continuity.

An integral represents the "net change in <what y represents> in <y unit> from $x = a$ <x unit> to $x = b$ <x unit>".

Area of equilateral triangle: $\frac{\sqrt{3}}{4}a^2$

Improper integrals: $\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$

Exponential differential shortcut: $\frac{dy}{dx} = ky \Rightarrow y = y_0 e^{kx}$

Propogated error: At x, if there is a potential error of $\pm\Delta x$, then the propogated error $\Delta y = y(x + \Delta x) - y(x)$.

This error can be approximated by $dy = f'(x)(\pm dx) \approx \Delta y$. Relative error $= \frac{dy}{y} = \frac{f'(x)dx}{f(x)}$.

Trig Identities	Log Rules	Physics	TI-83
$1 = \sin(x)^2 + \cos(x)^2$	$\log_b(xy) = \log_b(x) + \log_b(y)$	$v(t) = \frac{d}{dt}x(t)$	`nDeriv(f(X),X,value)`
$1 = \sec(x)^2 - \tan(x)^2$	$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$	$a(t) = \frac{d}{dt}v(t)$	`fnInt(f(X),X,a,b)`
$1 = \csc(x)^2 - \cot(x)^2$	$\log_b(x^y) = y \cdot \log_b(x)$	$W = Fd$	
$\cos(x)^2 = \frac{1 + \cos(2\theta)}{2}$	$\log_b(b^x) = x$	speed $= \lvert\vec{v}\rvert = \sqrt{v_x^2 + v_y^2}$	
$\sin(x)^2 = \frac{1 - \cos(2\theta)}{2}$	$\log_b(x) = \frac{\log_y(x)}{\log_y(b)}$	"speeding up" $\Rightarrow a(t), v(t) \neq 0$ and have like signs	
$\tan(x)^2 = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$	$\log_b(x) = \frac{1}{\log_x(b)}$	"slowing down" $\Rightarrow a(t), v(t) \neq 0$ and have opposite signs	

Logistic Function

A logistic function is of the differential form $\frac{dP}{dt} = kP(A - P)$, where $\lim_{t\to\infty} P(t) = A \Rightarrow A$ is the carrying capacity if a population is being modeled. For an initial value $(0, P_0)$ (the P-intercept), a solution to the differential is $P(t) = \frac{A}{1 + ce^{-kAt}}$, where $c = \frac{A - P_0}{P_0}$. $P(t)$ is growing fastest when $P = \frac{A}{2}$.

Euler's Method for Graphing a Solution to a Differential, Given an Initial Value

1. Begin at the given point, (x, y).
2. Use the differential equation to find $\frac{dy}{dx}$ at that point.
3. Find a new point, $(x + \Delta x, y + \Delta y)$, where $\Delta y = \frac{dy}{dx}\Delta x$ and Δx is a small value.
4. Go to step 2.

Use a negative Δx to plot the other half of the graph.